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Some mappings and fixed point theorems

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Abstract

It seems that a necessary condition for any $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mappings to have a fixed point is $\alpha+\beta+\gamma+\delta\geq 0$ or $\alpha+2\min\{\beta,\gamma\}+\delta\geq 0$. Already we know that the hypothesis is wrong. In this paper we show fixed point theorems in metric spaces. By using these results, we show fixed point theorems in Banach spaces and Hilbert spaces. Moreover our new fixed point theorems also do not need the both of the assumption $\alpha+\beta+\gamma+\delta\geq 0$ or $\alpha+2\min\{\beta,\gamma\}+\delta\geq 0$.

1 Introduction

Let E be a Banach space and let C be a non-empty subset of E. A mapping T from C into E is said to be widely more generalized hybrid [7] if there exist real numbers $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta$ and η such that

$$\begin{aligned}\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2\\ +\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2\leq 0\end{aligned}$$

for any $x,y\in C$. Such a mapping is called an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping. Let X be a metric space. A mapping T from X into itself is said to be widely more generalized hybrid [4] if there exist real numbers $\alpha,\beta,\gamma,\delta,\varepsilon$ and ζ such that

$$\alpha d(Tx,Ty)^2+\beta d(x,T)^2+\gamma d(Tx,y)^2+\delta d(x,y)^2+\varepsilon d(x,Tx)^2+\zeta d(y,Ty)^2\leq 0$$

for any $x,y\in X$. Such a mapping is called an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely more generalized hybrid mapping. The definition above introduced by Kawasaki and Takahashi in [10] in the case where E is a Hilbert space. In this paper we also use this definition in the case of Banach spaces. We obtained some fixed point theorems in the case of Hilbert spaces [3, 6, 10–12]. It seems that a necessary condition for any $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mappings to have a fixed point is $\alpha+\beta+\gamma+\delta\geq 0$ or $\alpha+2\min\{\beta,\gamma\}+\delta\geq 0$. Already we know that the hypothesis is wrong [7, 8]. In this paper we show fixed point theorems in metric spaces. By using these results, we show fixed point theorems in Banach spaces and Hilbert spaces. Moreover our new fixed point theorems also do not need the both of the assumption $\alpha+\beta+\gamma+\delta\geq 0$ or $\alpha+2\min\{\beta,\gamma\}+\delta\geq 0$.

2 Fixed point theorems

Let (X, d) be a metric space. Then

$$d(x,z)^2 - 2d(x,z)d(z,y) + d(z,y)^2 \leq d(x,y)^2 \leq d(x,z)^2 + 2d(x,z)d(z,y) + d(z,y)^2 \quad (2.1)$$

holds for any $x, y \in X$. Using this inequality, we obtain

Lemma 2.1 ([7]). *Let X be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely more generalized hybrid mapping from X into itself satisfying* $(\mathrm{B1})_m$, $(\mathrm{B2})_m$ *or* $(\mathrm{B3})_m$*:*

$(\mathrm{B1})_m$ $\alpha + \zeta + 2\min\{\beta, 0\} \geq 0$ *and* $\alpha + \delta + \varepsilon + \zeta + 4\min\{\beta, 0\} > 0$*;*

$(\mathrm{B2})_m$ $\alpha + \varepsilon + 2\min\{\gamma, 0\} \geq 0$ *and* $\alpha + \delta + \varepsilon + \zeta + 4\min\{\gamma, 0\} > 0$*;*

$(\mathrm{B3})_m$ $2\alpha + \varepsilon + \zeta + 2\min\{\beta + \gamma, 0\} \geq 0$ *and* $\alpha + \delta + \varepsilon + \zeta + 2\min\{\beta + \gamma, 0\} > 0$.

Then $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

By Lemma 2.1 we obtained the following theorem.

Theorem 2.1 ([7]). *Let X be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely more generalized hybrid mapping from X into itself satisfying one of* $(\mathrm{B1})_m$, $(\mathrm{B2})_m$ *and* $(\mathrm{B3})_m$, *and one of* $(\mathrm{M1})_m$, $(\mathrm{M2})_m$ *and* $(\mathrm{M3})_m$*:*

$(\mathrm{M1})_m$ $\alpha + \beta + \zeta > 0$*;*

$(\mathrm{M2})_m$ $\alpha + \gamma + \varepsilon > 0$*;*

$(\mathrm{M3})_m$ $2\alpha + \beta + \gamma + \varepsilon + \zeta > 0$.

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then the following hold:

(i) *T has a unique fixed point $u \in X$;*

(ii) *$u = \lim_{n\to\infty} T^n x$ for any $x \in X$.*

By Theorem 2.1 we obtain the following which the domain of mappings is also not required its convexity,

Theorem 2.2 ([7]). *Let H be a real Hilbert space, let C be a non-empty closed subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself satisfying one of* (B1), (B2) *and* (B3), *and one of* (M1), (M2) *and* (M3)*:*

(B1) $\alpha + \zeta + 2\min\{\beta, \eta\} \geq 0$ *and* $\alpha + \delta + \varepsilon + \zeta + 4\min\{\beta, \eta\} > 0$*;*

(B2) $\alpha + \varepsilon + 2\min\{\gamma, \eta\} \geq 0$ *and* $\alpha + \delta + \varepsilon + \zeta + 4\min\{\gamma, \eta\} > 0$*;*

(B3) $2\alpha + \varepsilon + \zeta + 2\min\{\beta + \gamma, 2\eta\} \geq 0$ *and* $\alpha + \delta + \varepsilon + \zeta + 2\min\{\beta + \gamma, 2\eta\} > 0$*;*

(M1) $\alpha+\beta+\zeta+\eta>0$;

(M2) $\alpha+\gamma+\varepsilon+\eta>0$;

(M3) $2\alpha+\beta+\gamma+\varepsilon+\zeta+2\eta>0$.

Then T has a fixed point. In particular, if $\alpha+\beta+\gamma+\delta>0$, then the following hold:

(i) *T has a unique fixed point $u \in C$;*

(ii) *$u=\lim_{n\to\infty} T^n x$ for any $x \in C$.*

Next we show another fixed point theorems. Using (2.1) we obtains the following lemmas.

Lemma 2.2 ([8]). *Let (X,d) be a metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely more generalized hybrid mapping from X into itself. Then the following hold:*

(1) *if $\alpha+\varepsilon+2\min\{\gamma,0\}>0$, then*

$$d(T^2x,Tx) \le \sqrt{A_1}d(Tx,x)$$

holds for any $x \in X$, where

$$A_1=\max\left\{-\frac{\delta+\zeta+2\min\{\gamma,0\}}{\alpha+\varepsilon+2\min\{\gamma,0\}},0\right\};$$

(2) *if $\alpha+\zeta+2\min\{\beta,0\}>0$, then*

$$d(T^2x,Tx) \le \sqrt{A_2}d(Tx,x)$$

holds for any $x \in C$, where

$$A_2=\max\left\{-\frac{\delta+\varepsilon+2\min\{\beta,0\}}{\alpha+\zeta+2\min\{\beta,0\}},0\right\};$$

(3) *if $2\alpha+\varepsilon+\zeta+2\min\{\beta+\gamma,0\}>0$, then*

$$d(T^2x,Tx) \le \sqrt{A_3}d(Tx,x)$$

holds for any $x \in C$, where

$$A_3=\max\left\{-\frac{2\delta+\varepsilon+\zeta+2\min\{\beta+\gamma,0\}}{2\alpha+\varepsilon+\zeta+2\min\{\beta+\gamma,0\}},0\right\}.$$

Lemma 2.3 ([8]). *Let (X,d) be a metric space and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$-widely more generalized hybrid mapping from X into itself. Then the following hold:*

(1) *if* $\alpha + 2\min\{\gamma, 0\} > 0$ *and* $\alpha + \varepsilon + 2\min\{\gamma, 0\} > 0$*, then*

$$d(T^3x, Tx) \le \sqrt{B_1}d(Tx, x)$$

holds for any $x \in C$*, where*

$$\begin{aligned} B_1 &= \max\left\{\max\left\{-\frac{\varepsilon}{\alpha + 2\min\{\gamma, 0\}}, 0\right\} A_1^2 \right. \\ &\quad \left. + \max\left\{-\frac{\beta + 2\min\{\delta, 0\}}{\alpha + 2\min\{\gamma, 0\}}, 0\right\} A_1 - \frac{\zeta + 2\min\{\gamma, 0\} + 2\min\{\delta, 0\}}{\alpha + 2\min\{\gamma, 0\}}, 0\right\}; \end{aligned}$$

(2) *if* $\alpha + 2\min\{\beta, 0\} > 0$ *and* $\alpha + \zeta + 2\min\{\beta, 0\} > 0$*, then*

$$d(T^3x, Tx) \le \sqrt{B_2}d(Tx, x)$$

holds for any $x \in C$*, where*

$$\begin{aligned} B_2 &= \max\left\{\max\left\{-\frac{\zeta}{\alpha + 2\min\{\beta, 0\}}, 0\right\} A_2^2 \right. \\ &\quad \left. + \max\left\{-\frac{\gamma + 2\min\{\delta, 0\}}{\alpha + 2\min\{\beta, 0\}}, 0\right\} A_2 - \frac{\varepsilon + 2\min\{\beta, 0\} + 2\min\{\delta, 0\}}{\alpha + 2\min\{\beta, 0\}}, 0\right\}; \end{aligned}$$

(3) *if* $\alpha + \min\{\beta + \gamma, 0\} > 0$ *and* $2\alpha + \varepsilon + \zeta + 2\min\{\beta + \gamma, 0\} > 0$*, then*

$$d(T^3x, Tx) \le \sqrt{B_3}d(Tx, x)$$

holds for any $x \in C$*, where*

$$\begin{aligned} B_3 &= \max\left\{\max\left\{-\frac{\varepsilon + \zeta}{2\alpha + 2\min\{\beta + \gamma, 0\}}, 0\right\} A_3^2 \right. \\ &\quad + \max\left\{-\frac{\beta + \gamma + 4\min\{\delta, 0\}}{2\alpha + 2\min\{\beta + \gamma, 0\}}, 0\right\} A_3 \\ &\quad \left. - \frac{\varepsilon + \zeta + 2\min\{\beta + \gamma, 0\} + 4\min\{\delta, 0\}}{2\alpha + 2\min\{\beta + \gamma, 0\}}, 0\right\}. \end{aligned}$$

Lemma 2.4 ([8]). *Let* (X, d) *be a metric space and let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-widely more generalized hybrid mapping from* X *into itself. Then the following hold:*

(1) *if* $\alpha + 2\min\{\gamma, 0\} > 0$ *and* $\alpha + \varepsilon + 2\min\{\gamma, 0\} > 0$*, then*

$$d(T^3x, T^2x) \le \sqrt{C_1}d(Tx, x)$$

holds for any $x \in C$*, where*

$$C_1 = \max\left\{-\frac{\gamma}{\alpha + \varepsilon}, 0\right\} B_1 + \max\left\{-\frac{\delta + \zeta}{\alpha + \varepsilon}, 0\right\} A_1;$$

(2) *if* $\alpha + 2\min\{\beta, 0\} > 0$ *and* $\alpha + \zeta + 2\min\{\beta, 0\} > 0$, *then*

$$d(T^3x, T^2x) \leq \sqrt{C_2} d(Tx, x)$$

holds for any $x \in C$, *where*

$$C_2 = \max\left\{-\frac{\beta}{\alpha+\zeta}, 0\right\} B_2 + \max\left\{-\frac{\delta+\varepsilon}{\alpha+\zeta}, 0\right\} A_2;$$

(3) *if* $\alpha + \min\{\beta+\gamma, 0\} > 0$ *and* $2\alpha + \varepsilon + \zeta + 2\min\{\beta+\gamma, 0\} > 0$, *then*

$$d(T^3x, T^2x) \leq \sqrt{C_3} d(Tx, x)$$

holds for any $x \in C$, *where*

$$C_3 = \max\left\{-\frac{\beta+\gamma}{2\alpha+\varepsilon+\zeta}, 0\right\} B_3 + \max\left\{-\frac{2\delta+\varepsilon+\zeta}{2\alpha+\varepsilon+\zeta}, 0\right\} A_3.$$

By Lemmas 2.2, 2.3 and 2.4 we obtain the following.

Theorem 2.3 ([8]). *Let E be a Banach space, let C be a non-empty closed subset of E and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, 0)$-widely more generalized hybrid mapping from C into itself. Suppose that one of the following conditions is satisfied:*

(1) $\alpha + 2\min\{\gamma, 0\} > 0$, $\alpha + \varepsilon + 2\min\{\gamma, 0\} > 0$ *and* $C_1 < 1$;

(2) $\alpha + 2\min\{\beta, 0\} > 0$, $\alpha + \zeta + 2\min\{\beta, 0\} > 0$ *and* $C_2 < 1$;

(3) $\alpha + \min\{\beta+\gamma, 0\} > 0$, $2\alpha + \varepsilon + \zeta + 2\min\{\beta+\gamma, 0\} > 0$ *and* $C_3 < 1$.

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then the following hold:

(i) *T has a unique fixed point $u \in C$;*

(ii) $u = \lim_{n\to\infty} T^n x$ *for any* $x \in C$.

By Theorem 2.3 we obtain the following which the domain of mappings is also not required its convexity. Let

$$\begin{aligned}
D_1 &= \max\left\{-\frac{\delta+\zeta+2\min\{\gamma,\eta\}}{\alpha+\varepsilon+2\min\{\gamma,\eta\}}, 0\right\}, \\
D_2 &= \max\left\{-\frac{\delta+\varepsilon+2\min\{\beta,\eta\}}{\alpha+\zeta+2\min\{\beta,\eta\}}, 0\right\}, \\
D_3 &= \max\left\{-\frac{2\delta+\varepsilon+\zeta+2\min\{\beta+\gamma,2\eta\}}{2\alpha+\varepsilon+\zeta+2\min\{\beta+\gamma,2\eta\}}, 0\right\}, \\
E_1 &= \max\left\{\max\left\{-\frac{\varepsilon+\eta}{\alpha-\eta+2\min\{\gamma,\eta\}}, 0\right\} D_1^2\right.
\end{aligned}$$

$$
\begin{aligned}
&+\max\left\{-\frac{\beta+\eta+2\min\{\delta,-\eta\}}{\alpha-\eta+2\min\{\gamma,\eta\}},0\right\} D_1-\frac{\zeta+\eta+2\min\{\gamma,\eta\}+2\min\{\delta,-\eta\}}{\alpha-\eta+2\min\{\gamma,\eta\}},0\Bigg\},\\
E_2 &= \max\left\{\max\left\{-\frac{\zeta+\eta}{\alpha-\eta+2\min\{\beta,\eta\}},0\right\} D_2^2\right.\\
&+\max\left\{-\frac{\gamma+\eta+2\min\{\delta,-\eta\}}{\alpha-\eta+2\min\{\beta,\eta\}},0\right\} D_2-\frac{\varepsilon+\eta+2\min\{\beta,\eta\}+2\min\{\delta,-\eta\}}{\alpha-\eta+2\min\{\beta,\eta\}},0\Bigg\},\\
E_3 &= \max\left\{\max\left\{-\frac{\varepsilon+\zeta+2\eta}{2\alpha-2\eta+2\min\{\beta+\gamma,2\eta\}},0\right\} D_3^2\right.\\
&+\max\left\{-\frac{\beta+\gamma+2\eta+4\min\{\delta,-\eta\}}{2\alpha-2\eta+2\min\{\beta+\gamma,2\eta\}},0\right\} D_3\\
&-\frac{\varepsilon+\zeta+2\eta+2\min\{\beta+\gamma,2\eta\}+4\min\{\delta,-\eta\}}{2\alpha-2\eta+2\min\{\beta+\gamma,2\eta\}},0\Bigg\},\\
F_1 &= \max\left\{-\frac{\gamma-\eta}{\alpha+\varepsilon+2\eta},0\right\} E_1+\max\left\{-\frac{\delta+\zeta+2\eta}{\alpha+\varepsilon+2\eta},0\right\} D_1,\\
F_2 &= \max\left\{-\frac{\beta-\eta}{\alpha+\zeta+2\eta},0\right\} E_2+\max\left\{-\frac{\delta+\varepsilon+2\eta}{\alpha+\zeta+2\eta},0\right\} D_2,\\
F_3 &= \max\left\{-\frac{\beta+\gamma-2\eta}{2\alpha+\varepsilon+\zeta+4\eta},0\right\} E_3+\max\left\{-\frac{2\delta+\varepsilon+\zeta+4\eta}{2\alpha+\varepsilon+\zeta+4\eta},0\right\} D_3.
\end{aligned}
$$

Theorem 2.4 ([8]). *Let H be a real Hilbert space, let C be a non-empty closed subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself. Suppose that one of the following conditions is satisfied:*

(1) $\alpha-\eta+2\min\{\gamma,\eta\}>0$, $\alpha+\varepsilon+2\min\{\gamma,\eta\}>0$ *and* $F_1<1$;

(2) $\alpha-\eta+2\min\{\beta,\eta\}>0$, $\alpha+\zeta+2\min\{\beta,\eta\}>0$ *and* $F_2<1$;

(3) $\alpha-\eta+\min\{\beta+\gamma,2\eta\}>0$, $2\alpha+\varepsilon+\zeta+2\min\{\beta+\gamma,2\eta\}>0$ *and* $F_3<1$.

Then T has a fixed point. In particular, if $\alpha+\beta+\gamma+\delta>0$, then the following hold:

(i) *T has a unique fixed point $u\in C$;*

(ii) *$u=\lim_{n\to\infty}T^n x$ for any $x\in C$.*

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